CYCLIC PALINDROMES

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Theorem. *If X, Y, Z are strings over some alphabet Σ such that XYZ, YZX, ZXY are all palindromes, then $|\{XYZ, YZX, ZXY\}| \leq 2$.*

Proof. We let

$$a \coloneqq |X|, \qquad b \coloneqq |XY|, \qquad n \coloneqq |XYZ|.$$

We let the dihedral group $D_{2n} \coloneqq \langle r, s \mid r^n = s^2 = (sr)^2 = 1 \rangle$ act on Σ^n by having r act as cyclic left rotation and having s act as reversal. By assumption,

$$\begin{aligned} r^0 \cdot XYZ &= XYZ, & r^a \cdot XYZ &= YZX, & r^b \cdot XYZ &= ZXY, \\ s \cdot XYZ &= XYZ, & s \cdot YZX &= YZX, & s \cdot ZXY &= ZXY. \end{aligned}$$

We define G to be the subgroup of D_{2n} generated by r^a and r^b. We define $g \coloneqq \gcd(a, b)$. By Bézout's identity, G is a cyclic group generated by r^g.

The action of D_{2n} on Σ^n restricts to a G-action on Σ^n. If we denote the G-orbit of XYZ by $G \cdot XYZ$, we have

$$\{XYZ, YZX, ZXY\} \subseteq G \cdot XYZ.$$

Let $\mathrm{Stab}_G(XYZ)$ denote the G-stabilizer of XYZ. We claim that r^{2a} and r^{2b} are contained in $\mathrm{Stab}_G(XYZ)$. Indeed,

$$r^{2a} \cdot XYZ = sr^{-a}sr^a \cdot XYZ = sr^{-a}s \cdot YZX = sr^{-a} \cdot YZX = s \cdot XYZ = XYZ$$

and similarly $r^{2b} \cdot XYZ = XYZ$. Again by Bézout, we also have $r^{2g} \in \mathrm{Stab}_G(XYZ)$, so the index of $\mathrm{Stab}_G(XYZ)$ in G is at most 2.

By the orbit-stabilizer theorem, we therefore have

$$|\{XYZ, YZX, ZXY\}| \leq |G \cdot XYZ| = \frac{|G|}{|\mathrm{Stab}_G(XYZ)|} \leq 2. \qquad \square$$